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Analytic expressions for integrals of products of spherical Bessel functions

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Abstract. Integrals of several spherical Bessel functions occur frequently in nuclear physics. They are difficult to evaluate using standard numerical techniques, because of their slowly decreasing oscillatory form. We derive an analytic expression for the infinite integral of three spherical Bessel functions. We then use this result, together with the closure relation for spherical Bessel functions, to show how in principle one can derive an analytic expression for the integral of any number of spherical Bessel functions. We demonstrate this by deriving an analytic expression for the integral of four spherical Bessel functions. As with all of these analytic formulae, our results require that all angular momenta corresponding to the spherical Bessel functions can be coupled together to give an overall scalar quantity and conserve parity. We discuss the numerical accuracy and stability of this procedure.

1. Introduction

Evaluation of nuclear reaction amplitudes or response functions in a partial-wave representation leads to integrals of the form

$$\int_0^\infty \mathcal{V}(r)\, r^2\, \mathrm{d}r \prod_{i=1}^{n} \chi_{l_i}(k_i r) \tag{1.1}$$

or to multiple integrals of similar structure. In this integral, $\mathcal{V}(r)$ is a nuclear interaction, and $\chi_l(kr)$ a bound or scattering-state radial wavefunction. The detailed structure of the integrals depends on the process under consideration, e.g., amplitude or response function; two-body or many-body final state; inelastic scattering, quasi-elastic scattering, rearrangement collision or multi-particle breakup.

Integrals of the form (1.1) can be attacked either in configuration or momentum space. Convergence tends to be more rapid in momentum space, but this advantage is more than counterbalanced by a serious unsolved problem: Coulomb distortions cannot at present be calculated in momentum space to better than 1 to 5% precision at small momentum transfers. There are situations in which the kinematics or the accuracy required render this disadvantage debatable. Nevertheless, for high-precision

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studies involving a wide kinematic range and a variety of observables, integrals of the sort considered here must be evaluated in configuration space.

Remarkably, after some fifty years of study of nuclear reactions and the nuclear response, numerical evaluation of integrals of the form (1.1) is a routine exercise only for processes involving two-body final states. In this paper, we discuss techniques applicable to multi-particle processes.

Suppose that the scattering states involved are expanded in a plane-wave basis:

$$\chi_l(kr) = \sum_i a_{l,i}\, j_l(k_i r) \tag{1.2}$$

where j_l is a spherical Bessel function for angular momentum l. Integrals of the form (1.1) are then expressed as linear combinations of basic integrals of one of two forms:

$$\int_0^\infty \mathrm{e}^{-\alpha r}\, r^2\, \mathrm{d}r \prod_{i=1}^{n} j_{\lambda_i}(k_i r) \tag{1.3}$$

or

$$\int_0^\infty r^2\, \mathrm{d}r \prod_{i=1}^{n} j_{\lambda_i}(k_i r). \tag{1.4}$$

In equation (1.3), the factor '$\mathrm{e}^{-\alpha r}$' represents a radial function with an asymptotic exponential falloff (which would appear in the integrals if they contain an exponential interaction factor $\mathcal{V}(r)$ or if one or more of the radial functions $\chi_{l_i}(k_i r)$ in (1.1) were bound).

Integrals of the form (1.3) or (1.4) have been extensively studied [1]†. Their difficulty stems from the presence of several rapidly oscillating factors with different periods. We can distinguish three different approaches for solving them:

(a) Angular-momentum methods wherein integrals of the form (1.4) are expressed as finite sums over products of $3j$ and $6j$ symbols, combinatorial factors and Legendre functions.

(b) Separation of integrals of the form (1.3) or (1.4) into a short-range and a long-range part, i.e.

$$\int_0^\infty = \int_0^R + \int_R^\infty . \tag{1.5}$$

The short-range part can be evaluated by standard numerical methods. The long-range part can be evaluated either by contour rotation or by reduction to a linear combination of known functions—sine, cosine or exponential integrals.

(c) Integration from zero to zero of the integrand, i.e.

$$\begin{aligned}\int_0^\infty &= \int_0^{\zeta_1} + \int_{\zeta_1}^{\zeta_2} + \ldots + \int_{\zeta_{n-1}}^{\zeta_n} + \ldots \\ &= a_1 + a_2 + \ldots + a_n + \ldots\end{aligned} \tag{1.6}$$

† References to previous work on integrals of products of Bessel functions, or to oscillatory functions which fall off as $1/r$ at large r, can be found in [1–4].

where the $\{\zeta_i\}$ are the ordered zeros of the integrand. Convergence accelerators can then be applied to the resulting series. If there is an exponential factor in the integral, the series $\{a_i\}$ converges rapidly and convergence acceleration will usually give ten or more figure accuracy with no more than five to ten terms. For integrals of the form (1.4), with no exponential factor, convergence is very slow (like $1/N$ for N terms); although convergence acceleration helps, the method is much less convenient than (a) or (b) above.

This paper deals with a number of aspects of the evaluation of integrals of the form (1.4), i.e. integrals of the form

$$I(\lambda_1\lambda_2...\lambda_n; k_1k_2...k_n) = \int_0^\infty r^2\,\mathrm{d}r \prod_{i=1}^{n} j_{\lambda_i}(k_i r). \tag{1.7}$$

Section 2 of this paper relates the work outlined here to previous studies of integrals of spherical Bessel functions. In subsequent sections of this paper we present the following results.

(i) In section 3 of this paper we derive a streamlined and correct variant of a published angular-momentum formula for the integral of the product of three spherical Bessel functions.

(ii) In section 4 we show how to extend this angular-momentum formula, in principle, to integrals of products of n spherical Bessel functions with $n > 3$; we give an explicit analytic formula for $n = 4$.

(iii) We study the numerical characteristics of the angular-momentum formulae, checking them and assessing their efficiency by comparison with the other two approaches (b) and (c) summarized earlier.

2. Relation to previous studies

The integral $I(\lambda_1\lambda_2\lambda_3; k_1k_2k_3)$ (in the notation of equation (1.7)) is the subject of several publications. An analytic expression for the integral of three Bessel functions is known [5]; however it involves calculation of a complicated hypergeometric function and is not suitable as a numerical quadrature formula [6]†. Explicit expressions of simpler forms have been derived in a variety of special cases.

Sawaguri and Tobocman [7] produce an expansion in modified harmonic-oscillator wavefunctions. Jackson and Maximon [1] give an expression in terms of associated Legendre functions. Anni and Taffara [8] derive a power series expansion of this integral, while Elbaz *et al* [9] give an expression in terms of Legendre polynomials. A summary of all of these formulae is given in a review article by Elbaz [10]. Recently, Davies and co-workers [2–4] have developed numerical methods for dealing with integrals which are oscillatory and very slowly convergent (in the notation of section 1, they use approach (b)).

Our formula for the integral of three spherical Bessel functions is essentially that of Elbaz [9]. Our results differ in the limit that the three momenta $\{\boldsymbol{k}_i\}$ are collinear. In order to show how this difference arises, and to discuss the ramifications of this difference, we present in section 3 the derivation of our formula for the integral of three

† The authors showed that under certain conditions the hypergeometric function of [5] simplifies and reduces to a finite sum.

spherical Bessel functions. In this section, we present some results for selected values of momenta and angular momenta, and we compare these with numerical results using approaches (b) and (c) as summarized in section 1. Stability criteria are discussed: our use of the vector spherical harmonic addition theorem makes it possible to encounter instabilities for certain values of the momenta. We show how such problems can be avoided. Derivation of our result requires a fair amount of recoupling algebra. The details of this algebra are deferred to the appendices.

In section 4, we show how the result for the integral of three spherical Bessel functions, together with the closure relation for spherical Bessel functions, can be used to derive an explicit formula for the integral of any number of spherical Bessel functions. This result is used to derive an analytic expression for the integral of four spherical Bessel functions, and our analytic formula is compared with numerical results for this integral.

All the angular-momentum formulae discussed here are valid only if (1) the partial-wave labels $\{\lambda_i\}$ can be coupled to zero resultant angular momentum. The first condition amounts to demanding that the quantity to be evaluated in deriving the integral formula is a scalar under rotation; and (2) the partial-wave labels have an *even* sum. The second condition arises from the condition that the amplitude conserves parity. Both of these conditions are satisfied for physical applications of the integrals.

3. Analytic expression for the integral of three spherical Bessel functions

In order to derive an analytic formula for the integral of three spherical Bessel functions, we begin with the expression for the δ-function as the integral over space of the plane wave,

$$(2\pi)^{-3}\int \exp[\mathrm{i}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\cdot\boldsymbol{r}]\mathrm{d}^3\boldsymbol{r} = \delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3). \tag{3.1}$$

If the left-hand side of this expression (LHS) is expanded in spherical Bessel functions and spherical harmonics, we obtain

$$\begin{aligned}\mathrm{LHS} = 8\sum_{\lambda_1\lambda_2\lambda_3} & i^{\lambda_1+\lambda_2+\lambda_3} Y^{\lambda_1 *}_{\mu_1}(\hat{k}_1) Y^{\lambda_2 *}_{\mu_2}(\hat{k}_2) Y^{\lambda_3}_{\mu_3}(\hat{k}_3)\\ & \times I(\lambda_1\lambda_2\lambda_3;k_1,k_2,k_3)\int \mathrm{d}\hat{r}\, Y^{\lambda_1}_{\mu_1}(\hat{r}) Y^{\lambda_2}_{\mu_2}(\hat{r}) Y^{\lambda_3 *}_{\mu_3}(\hat{r})\end{aligned} \tag{3.2}$$

where

$$I(\lambda_1\lambda_2\lambda_3;k_1,k_2,k_3) = \int_0^\infty r^2\mathrm{d}r\, j_{\lambda_1}(k_1 r) j_{\lambda_2}(k_2 r) j_{\lambda_3}(k_3 r) \tag{3.3}$$

is the integral we wish to express analytically. Using the well known relation for the integral of three spherical harmonics, equation (3.2) can be rewritten

$$\begin{aligned}[\mathrm{LHS}] = \frac{4}{\sqrt{\pi}}\sum_{\lambda_1\lambda_2\lambda_3} & i^{\lambda_1+\lambda_2-\lambda_3}[(2\lambda_1+1)(2\lambda_2+1)]^{1/2}\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ 0 & 0 & 0\end{pmatrix}\\ & \times I(\lambda_1\lambda_2\lambda_3;k_1,k_2,k_3) K(\lambda_1\,\lambda_2\,\lambda_3;\hat{k}_1,\hat{k}_2,\hat{k}_3)\end{aligned} \tag{3.4}$$

where the scalar function $K(\lambda_1\,\lambda_2\,\lambda_3;\hat{k}_1,\hat{k}_2,\hat{k}_3)$ is given by

$$K(\lambda_1\,\lambda_2\,\lambda_3;\hat{k}_1,\hat{k}_2,\hat{k}_3) = ([Y^{\lambda_1}(\hat{k}_1) \otimes Y^{\lambda_2}(\hat{k}_2)]^{\lambda_3} \cdot Y^{\lambda_3}(\hat{k}_3)) \tag{3.5}$$

and

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 0 & 0 \end{pmatrix}$$

is the Wigner 3-J symbol [11]. From this equation, it is clear that a necessary condition is that the three angular momenta $\{\lambda_i\}$ be coupled to an overall scalar, i.e. that $|\lambda_1 - \lambda_2| \le \lambda_3 \le \lambda_1 + \lambda_2$. For this reason, our result for this integral exists only if the three angular momenta obey the triangularity condition. In addition, the sum $\lambda_1 + \lambda_2 + \lambda_3$ must be even. The first restriction is a consequence of the conservation of angular momentum (the result must be a scalar under rotations), and the second is a consequence of parity conservation.

From the orthogonality relations for the scalar functions K,

$$\begin{aligned} \int K^*(\lambda_1\,\lambda_2\,\lambda_3;\hat{k}_1,\hat{k}_2,\hat{k}_3)\,K(\lambda_1{}'\,\lambda_2{}'\,\lambda_3{}';\hat{k}_1,\hat{k}_2,\hat{k}_3)\mathrm{d}\hat{k}_1\mathrm{d}\hat{k}_2\mathrm{d}\hat{k}_3 \\ = (2\lambda_3+1)\delta_{\lambda_1{}'\lambda_1}\delta_{\lambda_2{}'\lambda_2}\delta_{\lambda_3{}'\lambda_3} \end{aligned} \tag{3.6}$$

we can then derive an expression for the integral I

$$\begin{aligned} \frac{4}{\sqrt{\pi}} i^{\lambda_1+\lambda_2-\lambda_3}\left[(2\lambda_1+1)(2\lambda_2+1)\right]^{1/2}(2\lambda_3+1)\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 0 & 0 \end{pmatrix} I(\lambda_1\lambda_2\lambda_3;k_1,k_2,k_3) \\ = \int \delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)K^*(\lambda_1\,\lambda_2\,\lambda_3;\hat{k}_1,\hat{k}_2,\hat{k}_3)\mathrm{d}\hat{k}_1\mathrm{d}\hat{k}_2\mathrm{d}\hat{k}_3. \end{aligned} \tag{3.7}$$

Provided we can perform the integral on the right-hand side of equation (3.7), we will have an analytic expression for the integral of three spherical Bessel functions.

To evaluate this integral, note that the δ-function in equation (3.7) requires that $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$ form a closed triangle. This is geometrically possible only if the magnitudes satisfy

$$|k_1 - k_2| \le k_3 \le k_1 + k_2. \tag{3.8}$$

If this condition is satisfied, the quantity

$$\Delta = \frac{k_1^2 + k_2^2 - k_3^2}{2k_1\,k_2} \tag{3.9}$$

lies between ± 1 and is the cosine of the angle between $\hat{k}_1$ and $\hat{k}_2$ in the triangle formed by $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$.

Provided that the magnitudes of $\{\boldsymbol{k}_i\}$ satisfy equation (3.8), we may write

$$\begin{aligned} \delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) &= \frac{1}{k_3^2}\delta(k_3 - |\boldsymbol{k}_1+\boldsymbol{k}_2|)\,\delta(\hat{k}_3, -(\widehat{\boldsymbol{k}_1+\boldsymbol{k}_2})) \\ &= \frac{1}{k_1\,k_2\,k_3}\delta(x+\Delta)\,\delta(\hat{k}_3, -(\widehat{\boldsymbol{k}_1+\boldsymbol{k}_2})) \end{aligned} \tag{3.10}$$

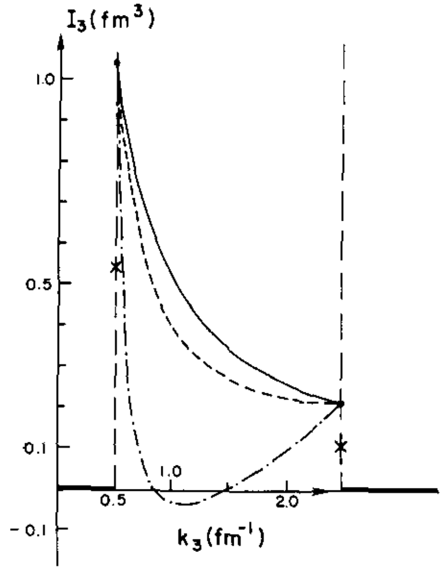

Figure 1. The integral $I(\lambda_1\lambda_2\lambda_3;k_1k_2k_3)$ of equation (3.3), in fm^3 for $k_1 = 1$ fm^{-1} and $k_2 = 1.5$ fm^{-1} as a function of k_3. With these values of k_1 and k_2 the integral vanishes for $k_3 < 0.5$ and $k_3 > 2.5$, and has jump discontinuities at the boundaries (denoted by xs). Full curve: $\lambda_1 = \lambda_2 = \lambda_3 = 0$; broken curve: $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$; chain curve: $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 3$. Note that (except for a phase factor as described in equation (3.23)) the value of the integral at the endpoints is independent of the angular momenta, for fixed values of the momenta.

where $x \equiv \cos(\hat{\boldsymbol{k}}_1, \hat{\boldsymbol{k}}_2)$†.

Substitution from equation (3.10) in equation (3.7) will be shown to yield an expression for the integral, equation (3.3), in the form of a Legendre series in Δ

$$I(\lambda_1\lambda_2\lambda_3; k_1k_2k_3) = \sum_l A_l P_l(\Delta) \tag{3.11}$$

valid only if $|\Delta| \leq 1$, i.e. if k_3 lies within the range of equation (3.8).

The integral in equation (3.11) is found by numerical quadrature to vanish identically if k_3 lies outside the range of equation (3.8); at the endpoints of the allowed interval, where $\boldsymbol{k}_1, \boldsymbol{k}_2$ and $\boldsymbol{k}_3$ are collinear, the integral has a value given by the average of its left- and right-hand limits. These properties are illustrated in figures 1–3. The Legendre series in equation (3.11), however, is a continuous function of Δ for all real Δ. Equation (3.11) as it stands cannot be true for $|\Delta| > 1$. Outside the allowed range of equation (3.8), the Legendre series must be cut-off and an additional factor of $\frac{1}{2}$ inserted for $|\Delta| = 1$. This may be accomplished by setting

$$I(\lambda_1\lambda_2\lambda_3; k_1k_2k_3) = \beta(\Delta) \sum_l A_l P_l(\Delta) \tag{3.12}$$

where

$$\beta(\Delta) = \vartheta(1 - \Delta)\vartheta(1 + \Delta) \tag{3.13}$$

† Equation (3.10) is valid for $k_3 \neq 0$. Methods appropriate when k_3 is small (or zero) are discussed at the end of section 3.

and $\vartheta(y)$ is the modified step function

$$\vartheta(y) = 0 \qquad y < 0$$
$$= \tfrac{1}{2} \qquad y = 0 \tag{3.14}$$
$$= 1 \qquad y > 0.$$

The coefficients A_l can now be evaluated for the geometrically allowed interval $-1 \leq \Delta \leq 1$. The closure relation for the Legendre polynomials,

$$\delta(x+\Delta) = \frac{1}{2}\sum_l (-1)^l\,(2l+1)\,P_l(x)P_l(\Delta) \tag{3.15}$$

can be rewritten, with the aid of the spherical-harmonic addition theorem, in the form

$$\delta(x+\Delta) = 2\pi\sum_l (-1)^l\,P_l(\Delta)(Y^l(\hat{k}_1)\cdot Y^l(\hat{k}_2)) \tag{3.16}$$

and substituted in equation (3.10). The result is

$$\delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) = \frac{2\pi}{k_1k_2k_3}\delta(\hat{k}_3, -(\widehat{\boldsymbol{k}_1+\boldsymbol{k}_2}))\sum_l (-1)^l P_l(\Delta)(Y^l(\hat{k}_1)\cdot Y^l(\hat{k}_2)). \tag{3.17}$$

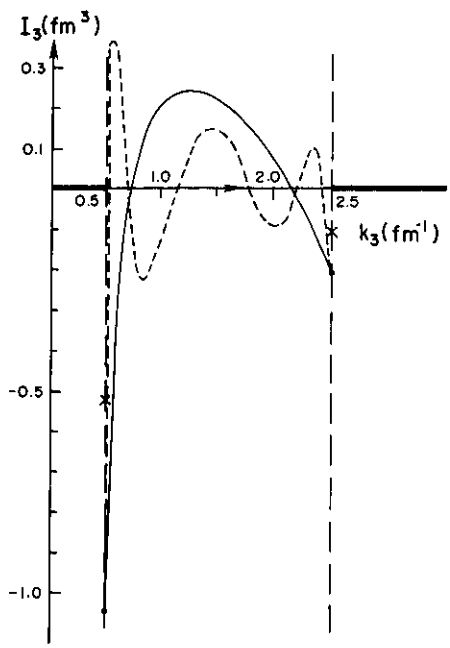

Figure 2. Same notation as figure 1. Full curve: $\lambda_1 = \lambda_2 = \lambda_3 = 2$; broken curve: $\lambda_1 = \lambda_2 = \lambda_3 = 6$.

Substituting equation (3.17) into equation (3.7) gives, including the factor $\beta(\Delta)$ from equation (3.12),

$$[\text{RHS of (3.7)}] = \frac{2\pi\beta(\Delta)}{k_1k_2k_3}\sum_l(-1)^l P_l(\Delta)\int \mathrm{d}\hat{k}_1\mathrm{d}\hat{k}_2(Y^l(\hat{k}_1)\cdot Y^l(\hat{k}_2)) \cdot K(\lambda_1\,\lambda_2\,\lambda_3;\hat{k}_1,\hat{k}_2,-(\widehat{k_1+k_2})) \tag{3.18}$$

valid for all real Δ. In equation (3.18) we have used the fact that K is a scalar product (see equation (3.5)) and hence K is real. To evaluate equation (3.18), we use the solid-harmonic addition theorem [12]

$$\text{if}\qquad \boldsymbol{r} = \boldsymbol{a}+\boldsymbol{b}$$

$$r^l Y^l_m(\hat{r}) = \sum_{k=0}^{l}\left(\frac{4\pi}{(2k+1)}\binom{2l+1}{2k}\right)^{1/2} a^{l-k}b^k\left[Y^{l-k}(\hat{a})\otimes Y^k(\hat{b})\right]^l_m . \tag{3.19}$$

In equation (3.19), $\binom{2l+1}{2k}$ is a binomial coefficient. Therefore we can write

$$Y^{\lambda_3}_{\mu_3}(\widehat{k_1+k_2}) = \left(\frac{k_1}{k_3}\right)^{\lambda_3}\sum_{L=0}^{\lambda_3}\left[\frac{4\pi}{(2L+1)}\binom{2\lambda_3+1}{2L}\right]^{1/2}\left(\frac{k_2}{k_1}\right)^L \cdot\left[Y^{\lambda_3-L}(\hat{k}_1)\otimes Y^L(\hat{k}_2)\right]^{\lambda_3}_{\mu_3}. \tag{3.20}$$

Substituting equation (3.20) into equation (3.18) and recoupling gives us our final analytic result. The recoupling and reduction are given in the appendices; after this algebra, we obtain

$$I(\lambda_1\lambda_2\lambda_3;k_1k_2k_3) = \frac{\pi\beta(\Delta)}{4k_1k_2k_3}i^{\lambda_1+\lambda_2-\lambda_3}(2\lambda_3+1)^{1/2}\left(\frac{k_1}{k_3}\right)^{\lambda_3} \cdot\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ 0 & 0 & 0\end{pmatrix}^{-1}\sum_{\mathcal{L}=0}^{\lambda_3}\binom{2\lambda_3}{2\mathcal{L}}^{1/2}\left(\frac{k_2}{k_1}\right)^{\mathcal{L}}\sum_l(2l+1)\begin{pmatrix}\lambda_1 & \lambda_3-\mathcal{L} & l\\ 0 & 0 & 0\end{pmatrix} \cdot\begin{pmatrix}\lambda_2 & \mathcal{L} & l\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mathcal{L} & \lambda_3-\mathcal{L} & l\end{Bmatrix}P_l(\Delta) \tag{3.21}$$

where the 6-J symbol is defined in (A.2).

Equation (3.21) is valid now for all real Δ, including values outside the limited range $-1 \le \Delta \le 1$, *with correct account taken of the jump discontinuities at* $\Delta = \pm 1$. The additional factor $\beta(\Delta)$ is present in the equation of Jackson and Maximon [1], but is absent in the formulae of Anni and Taffara [8] and Elbaz *et al* [9]. In our discussion we will demonstrate that this factor is necessary to give the correct numerical results for all values of the momenta. An easy way to demonstrate the correctness of our formula is to apply it to the case $\lambda_1 = \lambda_2 = \lambda_3 = 0$. In this case the spherical Bessel functions are just sine functions, and

$$
\begin{aligned}
I(0\,0\,0;k_1k_2k_3) &= \frac{1}{k_1\,k_2\,k_3}\int_0^\infty \sin(k_1r)\sin(k_2r)\sin(k_3r)\frac{\mathrm{d}r}{r} \\
&= \frac{\pi}{4k_1k_2k_3} \qquad \text{for } |k_1-k_2| < k_3 < k_1+k_2 \\
&= \frac{\pi}{8k_1k_2k_3} \qquad \text{for } k_3 = k_1+k_2 \text{ or } k_3 = |k_1-k_2| \\
&= 0 \qquad \text{otherwise.}
\end{aligned}
\tag{3.22}
$$

These integrals can be found in standard integral tables†; they agree with the results in equation (3.21) but not those of Elbaz *et al* nor those of Anni *et al* in the collinear limits, i.e. for $k_3 = k_1 + k_2$, or $k_3 = |k_1 - k_2|$. In figure 1, the full curve represents the integral of equation (3.22) with $k_1 = 1\ \mathrm{fm}^{-1}$ and $k_2 = 1.5\ \mathrm{fm}^{-1}$ as a function of k_3. The jump discontinuities at the endpoints are apparent.

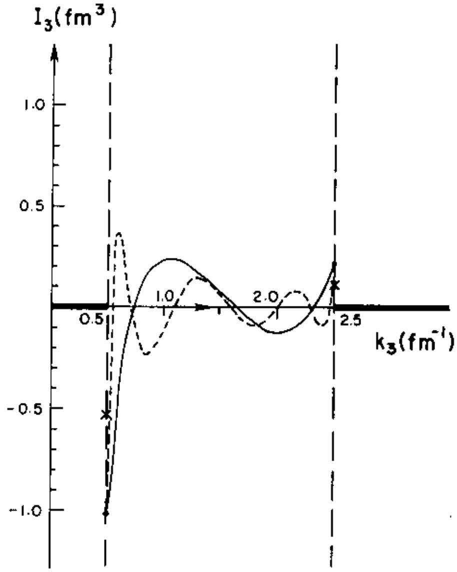

Figure 3. Same notation as figure 1. Full curve: $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$; broken curve: $\lambda_1 = 7$, $\lambda_2 = 6$, $\lambda_3 = 5$.

In fact, the magnitude of the integral (3.3), at the limits of the allowed range of values of k_3, *is independent of* $(\lambda_1, \lambda_2, \lambda_3)$,

$$
\begin{aligned}
&I(\lambda_1,\lambda_2,\lambda_3;k_1k_2k_3) \to \frac{\pi}{4k_1k_2k_3}(i)^{\lambda_1-\lambda_2-\lambda_3}[\mathrm{Sign}(k_1-k_2)]^{\lambda_3} \\
&\text{as } k_3 \to |k_1-k_2| \text{ from above} \\
&I(\lambda_1,\lambda_2,\lambda_3;k_1k_2k_3) \to \frac{\pi}{4k_1k_2k_3}(i)^{\lambda_1+\lambda_2-\lambda_3}
\end{aligned}
\tag{3.23}
$$

† See, e.g., equation (3.763.2) of [13].

as $k_3 \to k_1 + k_2$ from below.

At the endpoints, the limits are to be multiplied by the usual factor $\frac{1}{2}$. In figures 1–3 we plot results for $I(\lambda_1, \lambda_2, \lambda_3; k_1k_2k_3)$ for a variety of values of $\lambda_1, \lambda_2, \lambda_3$ (with $k_1 = 1\ \mathrm{fm}^{-1}$ and $k_2 = 1.5\ \mathrm{fm}^{-1}$). It is clear that the value of the integral at the endpoints is independent of the angular momenta except for the phase factors in equation (3.23). The integrals oscillate more and more rapidly near the endpoints as the angular momenta $\lambda_1, \lambda_2, \lambda_3$ increase.

It is straightforward to show from (3.21) that for a constant α

$$I(\lambda_1\lambda_2\lambda_3; (\alpha k_1), (\alpha k_2), (\alpha k_3)) = \frac{1}{\alpha^3} I(\lambda_1\lambda_2\lambda_3; k_1, k_2, k_3). \tag{3.24}$$

Equation (3.24) is known as the 'scaling formula' [2]. Table 1 shows some results for selected k and λ values, compared with results evaluated numerically, using approach (c) as outlined in section 1 of this paper. Note that because of the scaling formula, we could always choose one of the k values to be 1 and then rescale our answer using equation (3.24).

Table 1. The integral over 3 spherical Bessel functions evaluated analytically, equation (3.21), and numerically, using approach (c) of section 1, at selected k and λ values.

k_1	k_2	k_3	λ_1	λ_2	λ_3	Analytical result	Numerical result
1.0	2.0	1.5	0	0	0	0.2617993877991	0.2617993877992
1.0	2.0	1.5	0	1	1	0.2290744643243	0.2290744643243
1.0	2.0	1.5	1	1	0	0.1799870791119	0.1799870791119
1.0	2.0	1.5	3	2	1	0.1128754196419	0.1128754196419
1.0	2.0	1.5	4	4	4	−0.0456849264425	−0.0456849264425
1.0	5.0	5.5	0	0	0	0.0285599332145	0.0285599332145
1.0	5.0	5.5	4	4	4	−0.0080238752371	−0.0080238752369
1.0	1.05	0.06	0	0	0	12.46663751425	12.46663751425
1.0	1.05	0.06	4	4	4	−1.511653806929	−1.511653808097
1.0	1.02	0.03	0	0	0	25.6666066469754	25.6666066469758
1.0	1.02	0.03	4	4	4	−10.9755226231663	−10.9755226591813

It is evident from the definition of the integral I in equation (3.3) that the value of the integral is invariant under pairwise permutation of any of the ks (together with permutation of the corresponding λs). However, equation (3.21) *is not* obviously symmetric under permutation of momenta, or angular momenta. Is there some 'preferred' order of the momenta or angular momenta which will simplify the evaluation of equation (3.21)? The angular momentum sums on the right-hand side of equation (3.21) will contain the fewest terms if λ_3 is chosen as the *smallest* of the three angular momenta.

There is one exception to the general rule that λ_3 should be chosen as the smallest of the angular momenta. This occurs because the formula of equation (3.21) becomes numerically unstable in the limit where the momentum k_3 becomes much smaller than the other two momenta. This can be seen from the last four entries in table 1. In these cases the value k_3 is much smaller than k_1 and k_2. For these values there is a slight discrepancy between the analytic and numerical results; this discrepancy increases as k_3 decreases.

The origin of the instability is in the solid-harmonic addition theorem, equation (3.19), which produces an expression for solid harmonics of $\boldsymbol{r}$ in terms of those for $\boldsymbol{a}$ and $\boldsymbol{b}$, where $\boldsymbol{r} \equiv \boldsymbol{a} + \boldsymbol{b}$. For very small values of r (relative to a and b) and for $l \neq 0$, the left-hand side of equation (3.19) is very small while every term on the right-hand side is much larger. The final answer is obtained by almost complete cancellation of the large terms. This produces a loss of precision, and is a well known problem in any coupling scheme which involves use of the solid-harmonic addition theorem [14].

Table 2. Effect of permuting the ks on the intermediate results of equation (3.21), for $\lambda_1 = \lambda_2 = \lambda_3 = 4$. If k_3 is the smallest of the ks, there can be a tremendous cancellation between the intermediate results.

k_1	k_2	k_3	Intermediate result	Final result
1.0	1.05	0.06	956899.1520603	
			−2908695.703668	
			−1113452.336823	
			2403452.790684	
			3562274.570624	
			370821.3301599	
			−3206837.013293	
			−1227581.201348	
			1163116.899949	−1.511653806929
0.06	1.0	1.05	−0.0000204229484	
			0.0010450446734	
			−0.0008826159844	
			0.0434864286739	
			−0.0191557236181	
			−0.0054008143549	
			0.2902901870629	
			−0.2451711067769	
			−1.575844784825	−1.511653808097

Table 2 shows the problems which arise when k_3 is much smaller than k_1 and k_2. The intermediate results are displayed along with the final result. It is evident that the first six significant figures cancel in giving the final result. The solution to this problem is quite simple; if the momenta are permuted so that k_3 becomes the first argument (with corresponding permutation of the angular momenta), then the resulting sum is extremely stable and there is no strong cancellation, as is shown in the second half of table 2.

As a rule of thumb, whenever k_3 is more than an order of magnitude smaller than k_1 and k_2, it is best to permute the momenta so that the smallest momentum is the first, or second, argument in equation (3.21). For all other cases, setting λ_3 as the smallest of the angular momenta will optimize the sums carried out in equation (3.21).

Our result differs from that of Jackson and Maximon [2] as we use a different method for recoupling angular momenta. While we use the solid-harmonic addition theorem to express the spherical harmonics of one vector in terms of the other two vectors, Jackson and Maximon choose one of the three vectors as the z-axis and define the x–z plane by a second. Their final result is thus given in terms of associated Legendre functions. It is interesting to note that, since they avoid using the solid-harmonic addition theorem, Jackson and Maximon also avoid the possibility of strong cancellations whenever k_3 is very small.

4. Analytic expression for the integral of any number of spherical Bessel functions

In this section we show how the results of the previous section can be used to obtain an analytic result (in principle) for the integral of any number of spherical Bessel functions. To do this we employ a recursive procedure and make use of the spherical Bessel function closure relation

$$\delta(r-r') = \frac{2r^2}{\pi}\int_0^\infty k^2\,\mathrm{d}k\, j_L(kr)\, j_L(kr'). \tag{4.1}$$

Note that equation (4.1) is true for any value of the angular momentum L.

Let us show how this works in the case of the integral over four spherical Bessel functions, i.e.

$$I(l_1 l_2 l_3 l_4; k_1 k_2 k_3 k_4) = \int_0^\infty r^2\mathrm{d}r\, j_{l_1}(k_1 r)\, j_{l_2}(k_2 r)\, j_{l_3}(k_3 r)\, j_{l_4}(k_4 r). \tag{4.2}$$

Inserting the spherical Bessel function closure relation into equation (4.2) gives

$$\begin{aligned} I(l_1 l_2 l_3 l_4; k_1 k_2 k_3 k_4) &= \frac{2}{\pi}\int_0^\infty k^2\mathrm{d}k \int_0^\infty r^2\mathrm{d}r\, j_{l_1}(k_1 r)\, j_L(kr)\, j_{l_2}(k_2 r) \\ &\quad \times \int_0^\infty r'^2\mathrm{d}r'\, j_{l_3}(k_3 r')\, j_L(kr')\, j_{l_4}(k_4 r') \\ &= \frac{2}{\pi}\int_0^\infty k^2\mathrm{d}k I(l_1\, L\, l_2; k_1\, k\, k_2)\, I(l_3\, L\, l_4; k_3\, k\, k_4). \end{aligned} \tag{4.3}$$

Inserting the analytic form for the integral of three spherical Bessel functions (equation (3.21)) into equation (4.3) straightforwardly gives

$$\begin{aligned} I(\lambda_1\lambda_2\lambda_3\lambda_4; k_1 k_2 k_3 k_4) &= (-1)^L \frac{\pi i^{\lambda_1-\lambda_2+\lambda_3-\lambda_4}}{8k_1k_2k_3k_4}\left(\frac{k_1}{k_2}\right)^{\lambda_2}\left(\frac{k_3}{k_4}\right)^{\lambda_4}[(2\lambda_2+1)(2\lambda_4+1)]^{1/2} \\ &\cdot \begin{pmatrix}\lambda_1 & \lambda_2 & L\\ 0 & 0 & 0\end{pmatrix}^{-1}\begin{pmatrix}\lambda_3 & \lambda_4 & L\\ 0 & 0 & 0\end{pmatrix}^{-1}\sum_{\mathcal{L}=0}^{\lambda_2}\sum_{\mathcal{L}'=0}^{\lambda_4}\left(\binom{2\lambda_2}{2\mathcal{L}}\binom{2\lambda_4}{2\mathcal{L}'}\right)^{1/2} \\ &\cdot \sum_l\sum_{l'}(2l+1)(2l'+1)\begin{pmatrix}\lambda_1 & \lambda_2-\mathcal{L} & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\lambda_3 & \lambda_4-\mathcal{L}' & l'\\ 0 & 0 & 0\end{pmatrix} \\ &\cdot \begin{pmatrix}L & \mathcal{L} & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}L & \mathcal{L}' & l'\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}\lambda_1 & \lambda_2 & L\\ \mathcal{L} & l & \lambda_2-\mathcal{L}\end{Bmatrix}\begin{Bmatrix}\lambda_3 & \lambda_4 & L\\ \mathcal{L}' & l' & \lambda_4-\mathcal{L}'\end{Bmatrix} \\ &\cdot \left(\frac{J(k_1\, k_2\, k_3\, k_4; \mathcal{L}\mathcal{L}'\, l\, l')}{k_1^{\mathcal{L}} k_3^{\mathcal{L}'}}\right). \end{aligned} \tag{4.4}$$

Note that in equation (4.4), the left-hand side is *independent* of L. Consequently, this equation can be evaluated for any value of L, provided that it satisfies certain limits, i.e. $|\lambda_1-\lambda_2| \le L \le \lambda_1+\lambda_2$, $|\lambda_3-\lambda_4| \le L \le \lambda_3+\lambda_4$, and that both $\lambda_1+\lambda_2+L$ and $\lambda_3+\lambda_4+L$ are even. These limitations insure that the matrix element must be invariant under rotations, and that it satisfies parity conservation. Choosing L to have the smallest possible value minimizes the number of terms in equation (4.4).

In equation (4.4), all sums over angular momenta are finite. In this equation, the quantity $J(k_1\, k_2\, k_3\, k_4; \mathcal{L}\, \mathcal{L}'\, l\, l')$ is the integral over k in equation (4.3), defined by

$$J(k_1 k_2 k_3 k_4; \mathcal{L}\mathcal{L}'\, l\, l') = \int_0^\infty \mathrm{d}k\, \beta(\Delta_1)\, \beta(\Delta_3) k^{\mathcal{L}+\mathcal{L}'} P_l(\Delta_1)\, P_{l'}(\Delta_3). \quad (4.5)$$

In equation (4.5), the quantities Δ_1 and Δ_3 are defined by

$$\begin{aligned} \Delta_1 &= \frac{k^2 + k_1^2 - k_2^2}{2kk_1} \\ \Delta_3 &= \frac{k^2 + k_3^2 - k_4^2}{2kk_3} \end{aligned} \quad (4.6)$$

and the boundary function $\beta(x)$ is defined in equation (3.13). The function $P_l(x)$ in equation (4.5) is the Legendre polynomial of order l and argument x. Remember that the boundary function $\beta(x)$ vanishes if $|x| > 1$. Thus, for example the factor $\beta(\Delta_1)$ restricts the k-integration to values for which Δ_1 lies in the physical range $-1 \leq \Delta_1 \leq 1$ of the Legendre polynomials.

We can show that the boundary function $\beta(\Delta_1)$ restricts the values of k in the integral of equation (4.5) to

$$|k_2 - k_1| \leq k \leq k_1 + k_2. \quad (4.7)$$

Similarly, the boundary function $\beta(\Delta_3)$ restricts k to the interval

$$|k_4 - k_3| \leq k \leq k_3 + k_4. \quad (4.8)$$

For the purposes of this integral, we may replace the modified step-functions in equation (3.13) by unmodified Θ-functions. This is because we are interested in the area under this integral, and the fact that the modified Θ functions are discontinuous at one point makes no change in the area. It is then straightforward to show that the limits of integration in equation (4.5) are $\mathcal{K}_-$ to $\mathcal{K}_+$, where

$$\begin{aligned} \mathcal{K}_- &\equiv \mathrm{Max}(|k_2 - k_1|, |k_4 - k_3|) \\ \mathcal{K}_+ &\equiv \mathrm{Min}(k_1 + k_2, k_3 + k_4). \end{aligned} \quad (4.9)$$

The integral (4.5) vanishes unless $\mathcal{K}_+ > \mathcal{K}_-$. This condition would be violated only if the largest of the $\{k_i\}$ were larger than the sum of the other three; this is related to the requirement that the four vectors $\{\boldsymbol{k}_i\}$ in equation (4.2) satisfy the δ-function condition.

To evaluate the integral of equation (4.5), we expand the Legendre polynomials in power series in terms of the argument. Thus we use [15]

$$P_l(\Delta) = \sum_{s=0}^{E(l/2)} A_{l,s} \Delta^{l-2s} \quad (4.10)$$

Table 3. The integral over four spherical Bessel functions evaluated analytically, equation (4.4), and numerically, using approach (c) of section 1, at selected k and λ values.

k_1	k_2	k_3	k_4	λ_1	λ_2	λ_3	λ_4	Analytical result	Numerical result
1.0	1.0	1.0	1.0	0	0	0	0	0.785398163397	0.785398163398
1.0	1.0	1.0	1.0	0	5	2	3	0.003399992049	0.003399992049
1.0	1.0	1.0	1.0	4	4	4	4	0.153850922281	0.153850922281
1.0	2.0	2.0	3.0	0	0	0	0	0.065449846950	0.065449846950
1.0	2.0	2.0	3.0	0	5	2	3	0.004685892785	0.004685892785
1.0	2.0	2.0	3.0	4	4	4	4	0.001317986071	0.001317986071

where

$$A_{l,s} = \frac{(-1)^s (2l-2s)!}{2^l s!(l-s)!(l-2s)!} \tag{4.11}$$

and

$$\begin{aligned} E(l/2) &= \text{Integer part of } (l/2) \\ &= l/2 \qquad \text{for even } l \\ &= (l-1)/2 \qquad \text{for odd } l. \end{aligned} \tag{4.12}$$

Next we expand the arguments of the Legendre polynomials in equation (4.5) in terms of the integration variable k. Thus for example

$$\Delta_1 = \left(\frac{k_1^2 - k_2^2}{2k_1}\right)\frac{1}{k} + \frac{1}{2k_1}k. \tag{4.13}$$

Inserting equation (4.13) into equation (4.10) and using the binomial theorem gives

$$\Delta_1^{l-2s} = \left(\frac{k_1^2 - k_2^2}{2k_1}\right)^{l-2s} \sum_{\mu=0}^{l-2s} \binom{l-2s}{\mu} \left(\frac{1}{k_1^2 - k_2^2}\right)^{\mu} k^{2\mu+2s-l}. \tag{4.14}$$

This is a sum of even (odd) powers of k (when $(l-2s)$ is even (odd)) going from k^{2s-l} to k^{l-2s}.

With the analogous expansion for Δ_3, equation (4.5) becomes

$$\begin{aligned} J(k_1\,k_2\,k_3\,k_4;\mathcal{L}\mathcal{L}'\,l\,l') = \Theta(\mathcal{K}_+ - \mathcal{K}_-) \sum_{s=0}^{E(l/2)} \sum_{t=0}^{E(l'/2)} A_{l,s}A_{l',t} \left(\frac{k_1^2 - k_2^2}{2k_1}\right)^{l-2s} \\ \cdot \left(\frac{k_3^2 - k_4^2}{2k_3}\right)^{l'-2t} \sum_{\mu=0}^{l-2s} \binom{l-2s}{\mu} \left(\frac{1}{k_1^2 - k_2^2}\right)^{\mu} \sum_{\nu=0}^{l'-2t} \binom{l'-2t}{\nu} \left(\frac{1}{k_3^2 - k_4^2}\right)^{\nu} \\ \cdot \Upsilon_{\mathcal{L}+\mathcal{L}'+2\mu+2\nu+2s+2t-l-l'} \end{aligned} \tag{4.15}$$

where the value of the k-integral in equation (4.16) is given by

$$\Upsilon_\alpha \equiv \int_{\mathcal{K}_-}^{\mathcal{K}_+} k^\alpha \mathrm{d}k = \frac{\mathcal{K}_+{}^{\alpha+1} - \mathcal{K}_-{}^{\alpha+1}}{\alpha+1}. \tag{4.16}$$

This is certainly the case unless the exponent α in equation (4.16) is -1 (in which case the integral is logarithmic). However, from the triangular conditions on the angular momenta in equation (4.4), one can deduce that $\mathcal{L} + \mathcal{L}' + 2\mu + 2\nu + 2s + 2t - l - l'$ is always even.

Equations (4.4) and (4.15) give the analytic form for the integral of four spherical Bessel functions. In table 3 we compare our analytic results with numerical results for the same integrals. As can be seen we obtain at least seven-place accuracy with our sums. We have not found any regions of instability for our analytic result. However, these equations require sums over a large number of intermediate variables (eight). Such sums are extremely time-consuming; the computer time required increases exponentially as the angular momenta increase. For large values of the angular momenta in these integrals, it is considerably faster to evaluate equation (4.5) numerically. Since the integrals are finite they can be performed rapidly and accurately using Gauss–Legendre quadrature.

5. Conclusions

Integrals involving products of several spherical Bessel functions occur in many scattering problems in nuclear physics. For problems with a single continuum particle in both initial and final states, integrals involving three spherical Bessel functions are common. Such integrals are extremely difficult to evaluate numerically because of their poor convergence and oscillatory nature. For such problems several analytic expressions have been derived. For problems involving more particles (e.g. problems with one incident and two final-state continuum particles) one can encounter larger numbers of spherical Bessel functions. Knockout reactions such as $(\mathrm{e}, \mathrm{e}'\mathrm{p})$, $(\pi, 2\pi)$, or $(\mathrm{p}, \mathrm{p}'\pi)$ are often studied in nuclear physics and amplitudes for such processes may be evaluated by methods described in this paper.

We showed how an analytic expression for the integral of three spherical Bessel functions can be extended, using the closure relation for spherical Bessel functions, to produce an expression for the integral of any number of spherical Bessel functions. In section 4 we derived such an equation for four spherical Bessel functions (equation (4.4)). It involves a finite integral over intermediate momenta (for the product of n spherical Bessel functions one has to perform $n-3$ finite integrals).

The resulting integrals can be evaluated analytically by making power series expansions of the resulting Legendre functions. This is carried out for the integral of four spherical Bessel functions, and the analytic expression is given in equation (4.15). This expression is accurate and stable. However, even for small values of the angular momenta it involves lengthy computation due to the large number of sums to evaluate. For larger values of the angular momenta it is considerably faster to perform the integral (4.5) numerically.

We have not calculated integrals with more than four spherical Bessel functions. Although we could extend the methods outlined here, we know of no pressing physics questions which require these integrals. Furthermore, the number of sums in the resulting analytic expression will increase very rapidly with the number of Bessel functions. Finally, since $j_l(x) \rightarrow \sin(x - l\pi/2)/x$ for large x, integrals which contain large numbers of Bessel functions will exhibit much better convergence properties for large x and hence should be amenable to the numerical methods outlined in section 1.

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Appendix A. Summary of essential formulae involving angular momentum and spherical harmonics

States of angular momentum k and spherical tensors of rank k are, by definition, sets of $2k+1$ objects that transform under rotations like the spherical harmonics Y_q^k. The symbol $[\otimes]$ is used to indicate coupling of such objects to definite resultant angular momentum or rank, e.g.

$$[A^r \otimes B^s]_q^k = \sum_{\rho\sigma} (-1)^{s-r-q} (2k+1)^{\frac{1}{2}} \begin{pmatrix} r & s & k \\ \rho & \sigma & -q \end{pmatrix} A_\rho^r B_\sigma^s . \tag{A.1}$$

Recoupling of three angular momenta involves the $6j$ symbols defined by

$$\left[[A^r \otimes B^s]^t \otimes C^u\right]_q^k = \sum_{t'} [(2t+1)(2t'+1)]^{\frac{1}{2}} (-1)^{r+s+k+u} \begin{Bmatrix} r & s & t \\ u & k & t' \end{Bmatrix} \times \left[A^r \otimes [B^s \otimes C^u]^{t'}\right]_q^k . \tag{A.2}$$

Recoupling of four angular momenta to zero resultant also involves a $6j$ symbol (a degenerate $9j$ symbol) defined by

$$\left[[A^r \otimes B^s]^t \otimes [C^u \otimes D^v]^t\right]_0^0 = \sum_{t'} [(2t+1)(2t'+1)]^{\frac{1}{2}} (-1)^{s+u+t+t'} \begin{Bmatrix} r & s & t \\ v & u & t' \end{Bmatrix} \times [[A^r \otimes C^u]^{t'} \otimes [B^s \otimes D^v]^{t'}]_0^0 . \tag{A.3}$$

The product coupled to zero resultant $[\otimes]_0^0$ in equation (A.3) is related to the conventional scalar product by

$$\left(A^k \cdot B^k\right) = (-1)^k (2k+1)^{\frac{1}{2}} \left[A^k \otimes B^k\right]_0^0 . \tag{A.4}$$

The phase and normalization of the spherical harmonics are defined by

$$\begin{aligned} Y_m^{l*}(\hat{k}) &= (-1)^m Y_{-m}^l(\hat{k}) \\ \int Y_m^{l*}(\hat{k}) Y_{m'}^{l'}(\hat{k}) \mathrm{d}\hat{k} &= \delta_{ll'} \delta_{mm'} . \end{aligned} \tag{A.5}$$

Other useful properties include

$$\begin{aligned} Y_m^l(-\hat{k}) &= (-1)^l Y_m^l(\hat{k}) \\ Y_0^0(\hat{k}) &= \sqrt{1/4\pi} \end{aligned} \tag{A.6}$$

and the basic coupling theorem

$$\left[Y^r(\hat{k}) \otimes Y^s(\hat{k})\right]_q^t = \sqrt{1/4\pi} (-1)^t [(2r+1)(2s+1)]^{\frac{1}{2}} \begin{pmatrix} r & s & t \\ 0 & 0 & 0 \end{pmatrix} Y_q^t(\hat{k}) . \tag{A.7}$$

From equations (A.5) and (A.6) it follows that

$$\begin{aligned} \int Y^l_m(\hat{k})d\hat{k} &= \sqrt{4\pi}\delta_{l0}\delta_{m0} \\ \int \left(Y^l(\hat{k}_1)\cdot Y^l(\hat{k}_2)\right) d\hat{k}_1 d\hat{k}_2 &= 4\pi\delta_{l0}. \end{aligned} \tag{A.8}$$

The central result in our derivations of integrals of products of Bessel functions is the identity derived from equation (A.3) and the coupling theorem (A.7) for the zero-coupled product of four spherical harmonics of two independent unit vectors $\hat{k}_1, \hat{k}_2$. This identity is

$$\begin{aligned} &[[Y^{l_1}(\hat{k}_1)\otimes Y^{l_2}(\hat{k}_2)]^\lambda \otimes [Y^{l'_1}(\hat{k}_1)\otimes Y^{l'_2}(\hat{k}_2)]^\lambda]^0_0 \\ &\quad= \frac{1}{4\pi}(-1)^{l'_1+l'_2+\lambda}[(2l_1+1)(2l_2+1)(2l'_1+1)(2l'_2+1)]^{\frac{1}{2}} \\ &\quad\times \sum_\mu [(2\lambda+1)(2\mu+1)]^{\frac{1}{2}} \\ &\quad\cdot \begin{pmatrix} l_1 & l'_1 & \mu \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_2 & l'_2 & \mu \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & \lambda \\ l'_2 & l'_1 & \mu \end{Bmatrix} \left[Y^\mu(\hat{k}_1)\otimes Y^\mu(\hat{k}_2)\right]^0_0. \end{aligned} \tag{A.9}$$

When $\lambda = 0$ this reduces to

$$\begin{aligned} \left[\left[Y^l(\hat{k}_1)\otimes Y^l(\hat{k}_2)\right]^0 \otimes \left[Y^{l'}(\hat{k}_1)\otimes Y^{l'}(\hat{k}_2)\right]^0\right]^0_0 &= \frac{1}{4\pi}\sum_\mu [(2l+1)(2l'+1)(2\mu+1)]^{\frac{1}{2}} \\ &\times \begin{pmatrix} l & l' & \mu \\ 0 & 0 & 0 \end{pmatrix}^2 \left[Y^\mu(\hat{k}_1)\otimes Y^\mu(\hat{k}_2)\right]^0_0. \end{aligned} \tag{A.10}$$

Appendix B. Evaluation of the angular integral in equation (3.18)

The integral to be evaluated is

$$J = \int f(\hat{k}_1, \hat{k}_2)\, d\hat{k}_1\, d\hat{k}_2 \tag{B.1}$$

where $f(\hat{k}_1, \hat{k}_2)$ also depends on l, λ_1, λ_2, and λ_3 :

$$f(\hat{k}_1, \hat{k}_2) = (Y^l(\hat{k}_1)\cdot Y^l(\hat{k}_2))K(\lambda_1\,\lambda_2\,\lambda_3; \hat{k}_1\,\hat{k}_2\, - (\widehat{k_1+k_2})) \tag{B.2}$$

and the real scalar function K is defined in equation (3.5). As a scalar function of $\hat{k}_1, \hat{k}_2$, f can be expanded in the form

$$f(\hat{k}_1, \hat{k}_2) = \sum_r C_r(Y^r(\hat{k}_1)\cdot Y^r(\hat{k}_2)). \tag{B.3}$$

Given the expansion coefficients C_r, the angular integrations can be carried out using equation (A.8) of appendix A. The result is

$$J = 4\pi C_0. \tag{B.4}$$

To evaluate the desired integral we need only determine the $r = 0$ coefficient in the expansion (B.3).

First apply the recoupling identity (A.9) to the scalar function K. From the definition (3.5), using (A.4), (A.6) and the solid-harmonic expansion (3.19)

$$K = [\lambda_3]^{\frac{1}{2}}[[Y^{\lambda_1}(\hat{k}_1)\otimes Y^{\lambda_2}(\hat{k}_2)]^{\lambda_3}\otimes Y^{\lambda_3}(\widehat{k_1+k_2})]^0_0$$
$$= \sum_{L=0}^{\lambda_3}\alpha_L[[Y^{\lambda_1}(\hat{k}_1)\otimes Y^{\lambda_2}(\hat{k}_2)]^{\lambda_3}\otimes[Y^{\lambda_3-L}(\hat{k}_1)\otimes Y^{L}(\hat{k}_2)]^{\lambda_3}]^0_0 \tag{B.5}$$

where

$$\alpha_L = (2\lambda_3+1)^{\frac{1}{2}}\left(\frac{k_1}{k_3}\right)^{\lambda_3}\left[\frac{4\pi}{(2L+1)}\binom{2\lambda_3+1}{2L}\right]^{\frac{1}{2}}\left(\frac{k_2}{k_1}\right)^{L}. \tag{B.6}$$

Using the identity for binomial coefficients

$$\binom{2\lambda_3+1}{2L} = \frac{2\lambda_3+1}{2(\lambda_3-L)+1}\binom{2\lambda_3}{2L} \tag{B.7}$$

we obtain

$$\alpha_L = (2\lambda_3+1)\left(\frac{4\pi}{(2L+1)[2(\lambda_3-L)+1]}\right)^{\frac{1}{2}}\binom{2\lambda_3}{2L}^{\frac{1}{2}}\left(\frac{k_1}{k_3}\right)^{\lambda_3}\left(\frac{k_2}{k_1}\right)^{L}. \tag{B.8}$$

Using equation (A.9), the product of four spherical harmonics in equation (B.5) reduces to

$$K = \sum_s \beta_{sL}\left[Y^s(\hat{k}_1)\otimes Y^s(\hat{k}_2)\right]^0_0 \tag{B.9}$$

with

$$\beta_{sL} \equiv \frac{[(2\lambda_1+1)(2\lambda_2+1)(2\lambda_3+1)(2\lambda_3-2L+1)(2L+1)(2s+1)]^{\frac{1}{2}}}{4\pi} \cdot\begin{pmatrix}\lambda_1 & \lambda_3-L & s\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\lambda_2 & L & s\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ L & \lambda_3-L & s\end{Bmatrix} \tag{B.10}$$

hence

$$f(\hat{k}_1,\hat{k}_2) = (-1)^l(2l+1)^{\frac{1}{2}}\sum_L\sum_s \alpha_L\beta_{sL} \cdot[[Y^l(\hat{k}_1)\otimes Y^l(\hat{k}_2)]^0\otimes[Y^s(\hat{k}_1)\otimes Y^s(\hat{k}_2)]^0]^0_0 \tag{B.11}$$

where we have used equation (A.4).

The zero-coupled product of four spherical harmonics in equation (B.11) can now be reduced using equation (A.10) of appendix A. The result is

$$f(\hat{k}_1,\hat{k}_2) = \frac{1}{4\pi}\sum_r\sum_{L,s}\alpha_L\beta_{sL}(-1)^s(2s+1)^{\frac{1}{2}}(2l+1)\begin{pmatrix}l & s & r\\ 0 & 0 & 0\end{pmatrix}^2\left(Y^r(\hat{k}_1)\cdot Y^r(\hat{k}_2)\right). \tag{B.12}$$

This is an expansion of the form (B.3). Only the $r = 0$ coefficient is needed. In this case

$$(2s+1)^{\frac{1}{2}}(2l+1)\begin{pmatrix} l & s & r \\ 0 & 0 & 0 \end{pmatrix}^2 \Big|_{r=0} = (2l+1)(2s+1)^{\frac{1}{2}}\frac{\delta_{ls}}{(2l+1)} = (2l+1)^{\frac{1}{2}}\delta_{ls}. \tag{B.13}$$

Therefore (see equation (B.3))

$$C_0 = (-1)^l\,(2l+1)^{\frac{1}{2}}\frac{1}{4\pi}\sum_L \alpha_L\beta_{lL} \tag{B.14}$$

where α_L and β_{sL} are given by equations (B.8) and (B.10). Substitution into equation (B.4) yields the desired expression for the angular integral (B.1);

$$J = \sqrt{1/4\pi}(2\lambda_3+1)\left(\frac{k_1}{k_3}\right)^{\lambda_3}[(2\lambda_1+1)(2\lambda_2+1)(2\lambda_3+1)]^{\frac{1}{2}}(-1)^l$$
$$\times\sum_{L=0}^{\lambda_3}(2l+1)\left(\frac{k_2}{k_1}\right)^L\binom{2\lambda_3}{2L}^{\frac{1}{2}}\begin{pmatrix}\lambda_1 & \lambda_3-L & l \\ 0 & 0 & 0\end{pmatrix}$$
$$\times\begin{pmatrix}\lambda_2 & L & l \\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3 \\ L & \lambda_3-L & l\end{Bmatrix}. \tag{B.15}$$

Substitution from equation (B.15) in (3.18), followed by use of equations (3.7) and (3.12) yields the final expression (3.21) for the integral of a product of three spherical Bessel functions.

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